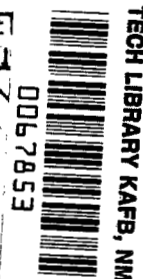


**NASA  
Technical  
Paper  
2216**

**August 1983**

NASA  
TP  
2216  
c.1

LOAN COPY: RET  
AFWL TECHNICAL  
KIRTLAND AFB, N



# **Analysis of Random Signal Combinations for Spacecraft Pointing Stability**

**Leonard Howell**

**NASA**



25th Anniversary  
1958-1983

**NASA  
Technical  
Paper  
2216**

**1983**

TECH LIBRARY KAFB, NM



0067853

# **Analysis of Random Signal Combinations for Spacecraft Pointing Stability**

**Leonard Howell**

*George C. Marshall Space Flight Center  
Marshall Space Flight Center, Alabama*

**NASA**  
National Aeronautics  
and Space Administration  
  
Scientific and Technical  
Information Branch



## TABLE OF CONTENTS

	Page
I. INTRODUCTION .....	1
II. METHODS .....	1
III. APPLICATION .....	2
A. Gaussian Noise on Exponential Curve .....	2
B. Gaussian Noise on Cosine Wave .....	4
C. Gaussian Noise on Damped Sine Wave .....	6
D. Gamma on Damped Sine Wave .....	8
IV. CONCLUSIONS .....	8
REFERENCES .....	10
APPENDIX A – STEREO VISUALIZATION WITHOUT OPTICAL AIDS (Cross-Eyed Stereo) .....	11

## LIST OF ILLUSTRATIONS

Figure	Title	Page
1a.	Rate gyroscope noise on image drift (smooth curve) versus time. ....	3
1b.	Probability density function of Gaussian noise on exponential function. ....	3
2a.	Gaussian noise ( $\sigma_N = 0.2$ ) on cosine wave (amplitude 2). ....	4
2b.	Family of distributions for Gaussian noise ( $\sigma_N = 0.2$ ) on cosine wave. Amplitude between 1 (narrowest bi-modal curve) and 4 (widest). ....	5
3a.	Gaussian noise on damped sine wave. ....	6
3b.	Density function for Gaussian noise on a damped sine wave. ....	7
3c.	Stereo picture of family of distributions for Gaussian noise on damped sine wave, with $\sigma_N = 0.2$ , $\beta = 0.3$ , and the amplitude varying between 1 (foremost $U_Z(Z)$ ) and 4 in the Y-axis. ....	7
4a.	Integrand in equation (15) for gamma distribution on damped sine. The X-axis represents the variable $t$ and the Y-axis the variable $Z$ . ....	9
4b.	Probability density function for gamma noise on damped sine wave. ....	9
A-1.	Stereo projection. ....	12
A-2.	Stereo reconstruction by cross-eyed viewing. ....	13
A-3.	Projection geometry. ....	13

## TECHNICAL PAPER

# ANALYSIS OF RANDOM SIGNAL COMBINATIONS FOR SPACECRAFT POINTING STABILITY

## I. INTRODUCTION

Design and performance criteria for space vehicle pointing systems must frequently incorporate combinations of random signals. This often involves a white noise process superimposed on a time varying error profile. For example, as the Space Telescope<sup>1</sup> emerges from behind the shadow of the Earth, solar heating is expected to produce a structural deformation resulting in an image error which can be adequately described by an exponential process. Independent and additive to this error is the rate gyroscope noise which is assumed to follow a normal distribution.

Because the control system designer needs some criteria to assess the impact of these disturbances on the performance of the system, the probability distribution of the sum of these two effects is required. Typically, the RMS (Root Mean Square) is used as a measure of performance. This, among other important information, may be obtained from the probability density function of the sum of these two disturbances.

Since the error profile is a known function of time, the statistical properties of the combined errors for a specified time  $t$  can easily be evaluated. However, this is, in general, not representative of typical signal analysis. For example, a frequently used technique picks the maximum of the known error profile and then adds to that maximum another two or three standard deviations of the superimposed noise distribution. This is clearly an overly conservative method for most applications and puts an unrealistic demand on the control system designer.

The approach to be used here is to imagine the system being observed at random times. What then is the statistical behavior of the combined observed errors over the time domain of the error profile?

This is equivalent to considering time to be a uniformly distributed random variable, with the goal now being the determination of the noise and error profile, the latter now being treated as a random variable as well.

## II. METHODS

Let  $T$  be a random variable with a density function given by

$$f_T(t) = \begin{cases} 1/t & , \quad 0 < t \leq \tau \\ 0 & \text{elsewhere} \end{cases} \quad (1)$$

---

1. Space Telescope is a pioneering observatory scheduled for in-orbit operation in mid-1986.

The error profile random variable is denoted by  $X$ , where  $X = h(T)$ . The noise variable is represented by  $Y$  with density function  $g_Y(\cdot)$ . Let  $u_Z(\cdot)$  denote the probability density function for  $Z = X + Y$ .

By the method of convolutions, the resultant probability density function of the random variable  $Z$  is given by

$$u_Z(z) = \int_{\Omega} g_Y[z - h(t)] f_T(t) dt \quad , \quad (2)$$

where the integral is over the set  $\Omega = \{ t: z-h(t) \in \text{Domain}(y) \}$ .

As an alternative method, one may first obtain the characteristic function of the random variable  $Z$  as

$$\phi_Z(\xi) = \phi_Y(\xi) \phi_X(\xi) \quad , \quad (3)$$

by independence of  $X$  and  $Y$ .

Assuming  $\phi_Z(\xi)$  to be Lebesgue integrable over the entire line, we have:

$$u_Z(z) = 1/2\pi \int_{-\infty}^{\infty} e^{-i\xi z} \phi_Z(\xi) d\xi \quad , \quad (4)$$

as the desired probability density function of the random variable  $Z$  [1]. These methods will be illustrated in the following section.

### III. APPLICATION

#### A. Gaussian Noise on Exponential Curve

Thermal models predict that solar heating of the Space Telescope will produce a structural deformation which will induce an image drift (arc sec) described by the equation

$$x = h(t) = 0.0037(1 - e^{-t/4}) \quad , \quad 0 < t \leq 24 \text{ hours} \quad . \quad (5)$$

The rate gyroscope, used for attitude determination, has an associated noise profile  $Y$  which is assumed to be Gaussian with zero mean and a standard deviation of 0.0047 arc sec (Fig. 1a).

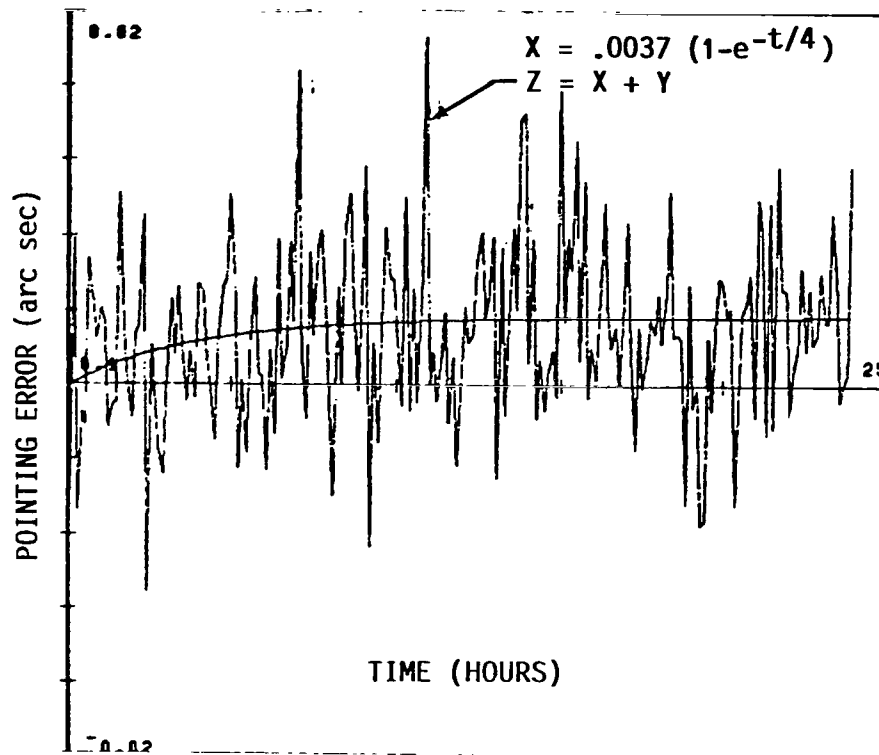


Figure 1a. Rate gyroscope noise on image drift (smooth curve) versus time.

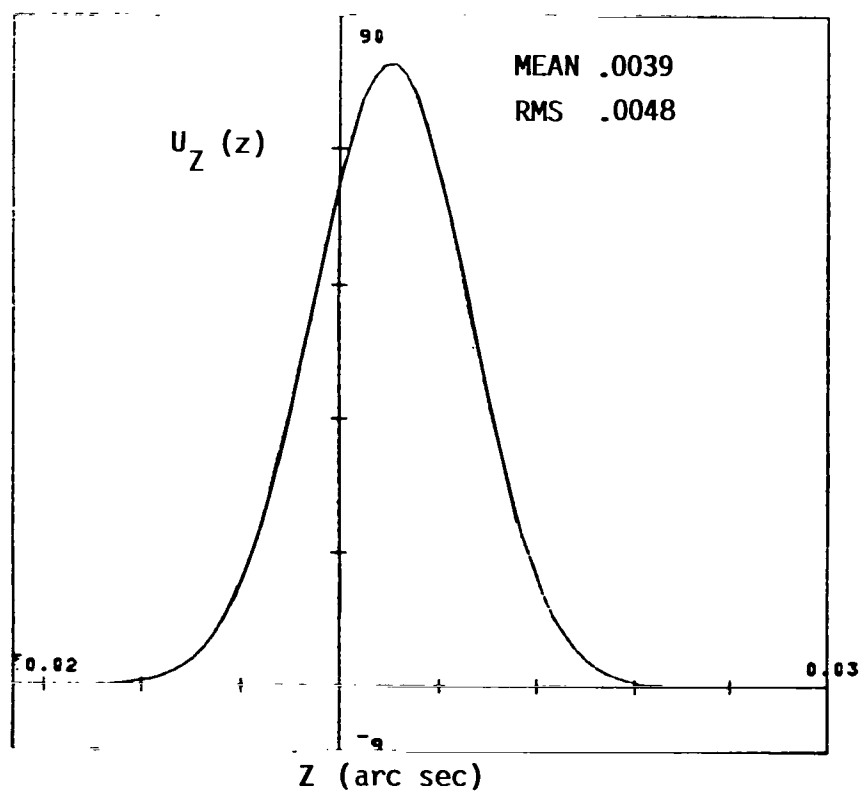


Figure 1b. Probability density function of Gaussian noise on exponential function.



The resultant density of the sum of the two is given by:

$$u_Z(z) = \frac{1}{T\sqrt{2\pi}\sigma_N} \int_0^{24} \frac{e^{-1}}{e^{2\sigma_N^2}} [z - 0.0037(1 - e^{-t/4})]^2 dt \quad . \quad (6)$$

Numerical integration gives  $u_Z(z)$  as depicted in Figure 1b.

### B. Gaussian Noise on Cosine Wave

Imbalances in the Space Telescope reaction wheels result in a sinusoidal image error of constant amplitude  $A$ . Rate gyroscope noise, combined with guide star photon fluctuation counts, is assumed to be superimposed on this sinusoidal error profile as Gaussian white noise (Fig. 2a). Thus, we have  $X = h(T)$ , where

$$h(t) = A \cos t \quad 0 \leq t \leq 2\pi \quad . \quad (7)$$

Using the method of characteristic functions, we have

$$\phi_{h(t)}(\xi) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\xi A \cos t} dt \quad , \quad (8)$$

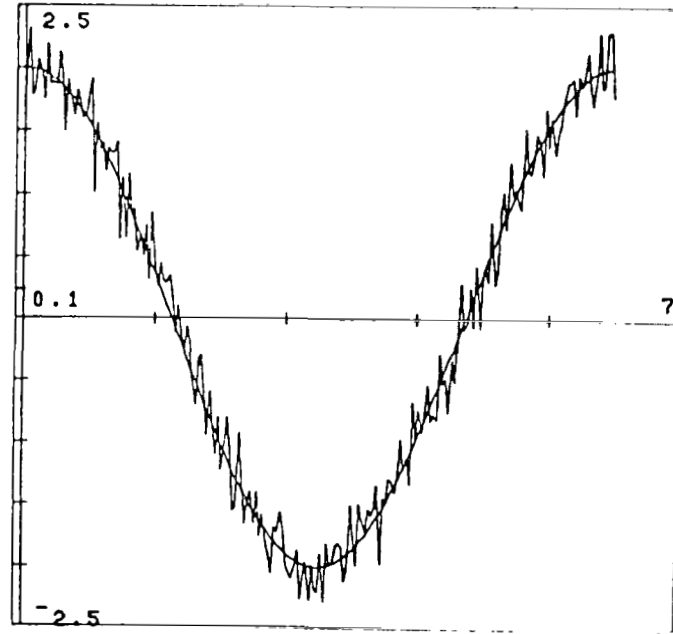


Figure 2a. Gaussian noise ( $\sigma_N = 0.2$ ) on cosine wave (amplitude 2).

which yields:

$$\phi_{h(t)}(\xi) = \frac{1}{\pi} \int_0^{\pi} \cos(A \xi \cos t) dt = J_0(A\xi) , \quad (9)$$

where  $J_0$  is Bessell's equation of order zero. Hence, for Gaussian noise  $Y$  with characteristic function

$$\phi_Y(\xi) = e^{-\frac{\xi^2 \sigma_N^2}{2}} , \quad (10)$$

and using the relation

$$J_n(\xi) = \frac{i^{-n}}{\pi} \int_0^{\pi} e^{i\xi \cos \theta} \cos(n\theta) d\theta \quad [2] ,$$

equation (4) yields

$$u_Z(z) = \frac{1}{\pi} \frac{1}{\sqrt{2\pi \sigma_N^2}} \int_0^{\pi} e^{-\frac{1}{2\sigma_N^2} (z - A \cos t)^2} dt , \quad (11)$$

which agrees with the results obtained by the method of convolutions. Figure 2b depicts the family of distributions as a result of varying  $A$  between one and four, keeping  $\sigma_N = 0.2$  fixed.

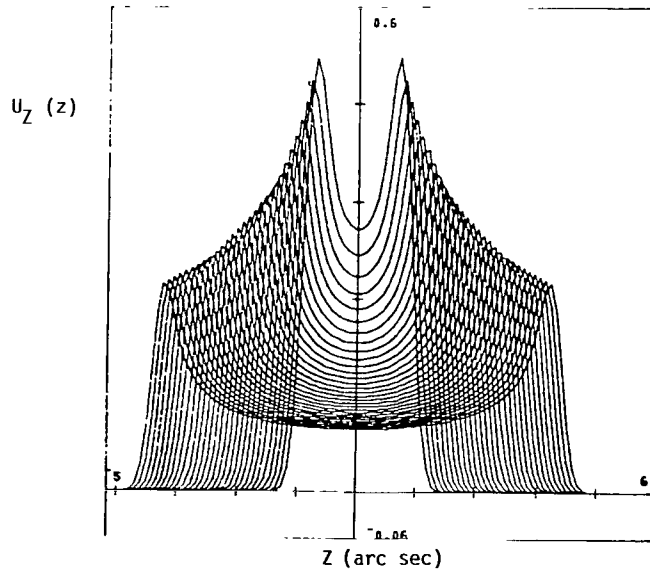


Figure 2b. Family of distributions for Gaussian noise ( $\sigma_n = 0.2$ ) on cosine wave. Amplitude between 1 (narrowest bi-modal curve) and 4 (widest).

### C. Gaussian on Damped Sine Wave

Vibrations of solar masts of the Space Shuttle Orbiter payloads are expected to produce a damped sinusoidal pointing disturbance for on-board sensors. Independent and additive to this error is a noise source which follows the normal distribution with zero mean and standard deviation  $\sigma_N$ . Figure 3a depicts Gaussian noise ( $\sigma_N = 0.2$ ) on a damped sine function of amplitude 2.5 and damping coefficient of 0.3, shown for four time constants. Hence, the resultant density function is given by

$$u_Z(z) = \frac{1}{T} \left( \frac{1}{2\pi \sigma_N^2} \right)^{1/2} \int_0^T e^{-\frac{1}{2\sigma_N^2} (z - A e^{-\beta t} \sin t)^2} dt, \quad (12)$$

where T extends over four time constants.

The density function  $u_Z(z)$  in Figure 3b is for the above parameters. The family of distributions obtained for  $\sigma_N = 0.2$ ,  $\beta = 0.3$ , and  $1 \leq A \leq 4$  is shown by the stereo pair<sup>2</sup> in Figure 3c.

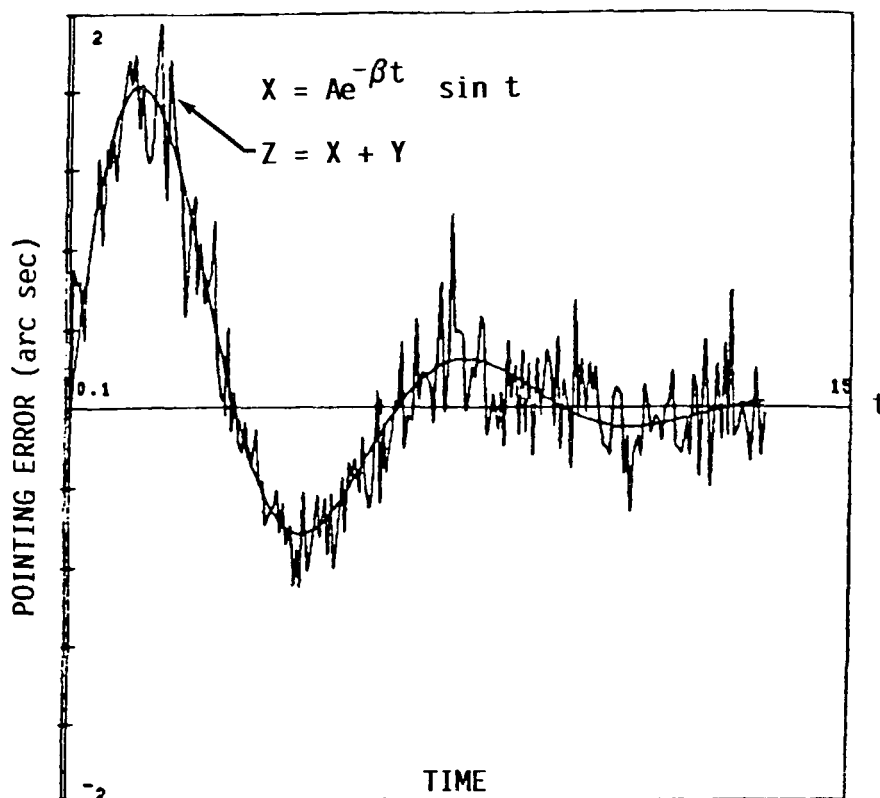


Figure 3a. Gaussian noise on damped sine wave.

2. See Appendix A, Stereo Visualization Without Optical Aids.

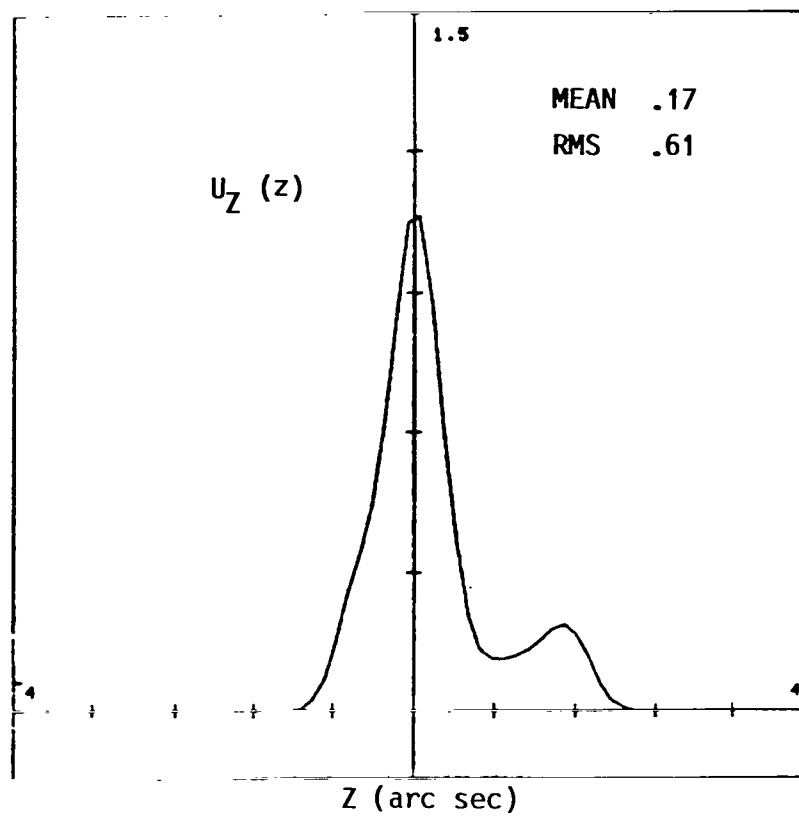


Figure 3b. Density function for Gaussian noise on a damped sine wave.

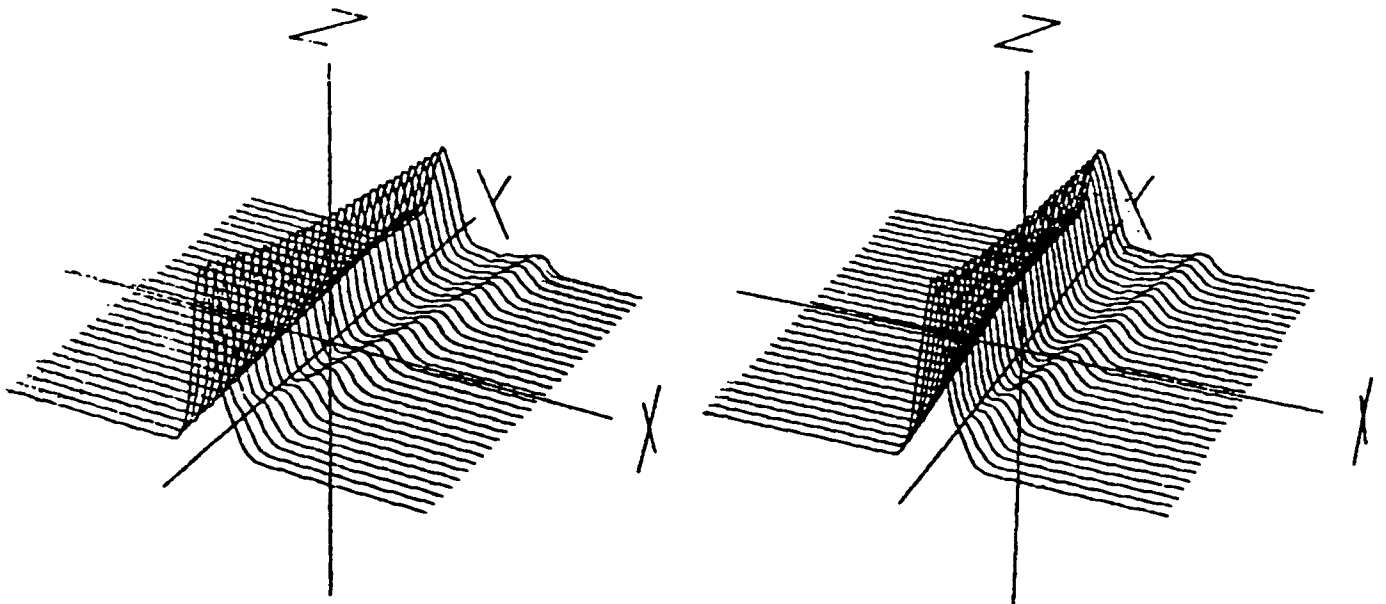


Figure 3c. Stereo picture of family of distributions for Gaussian noise on damped sine wave, with  $\sigma_N = 0.2$ ,  $\beta = 0.3$ , and the amplitude varying between 1 (foremost  $U_Z(Z)$ ) and 4 in the Y-axis.

#### D. Gamma on Damped Sine Wave

Quite often the noise source is non-Gaussian and the gamma distribution, being very flexible, may be suitable for many processes where the random variable takes on values greater than zero. If the error signal is a damped sine wave given by

$$h(t) = A e^{-\beta t} \sin t \quad , \quad (13)$$

and the noise distribution is gamma with density function given by

$$g_y(y) = \frac{\alpha^n}{\Gamma(n)} y^{n-1} e^{-\alpha y} \quad , \quad \text{where } \alpha, n, \text{ and } y > 0 \quad , \quad (14)$$

we obtain the following density function for  $Z = X + Y$ :

$$u(z) = \frac{\alpha^n}{\Gamma(n)} \frac{1}{T} \int_0^T I(Z - A e^{-\beta t} \sin t) (Z - A e^{-\beta t} \sin t)^{n-1} e^{-\alpha(Z - A e^{-\beta t} \sin t)} dt \quad , \quad (15)$$

where the indicator function  $I$  is used and defined by

$$I(x) = \begin{cases} 0 & X \leq 0 \\ 1 & X > 0 \end{cases} \quad . \quad (16)$$

Figure 4a depicts the integrand in equation (15) with  $\alpha = 1$ ,  $n = 2$ ,  $\beta = 0.5$ , and  $A = 4$ . Figure 4b shows the resultant density function.

#### IV. CONCLUSIONS

The control system designer of spacecraft pointing systems must frequently incorporate combinations of random signals and their effects in the overall control design. Traditionally, when the probability distribution of the combined signals is unknown, the designer will use overly conservative measures of these effects which place restrictive bounds on the control design.

Two methods are presented for characterizing the statistical behavior of a random signal superimposed on an error profile. Each assumes that the two processes are independent and additive. The first method computes the probability density function of the combined error signals by the use of convolutions and is usually amenable to numerical methods. However, the second method, which uses characteristic functions, may in some cases be more tractable. Thus, one will find both methods useful and, in some cases, it may be enlightening to use one method as a check on the other.

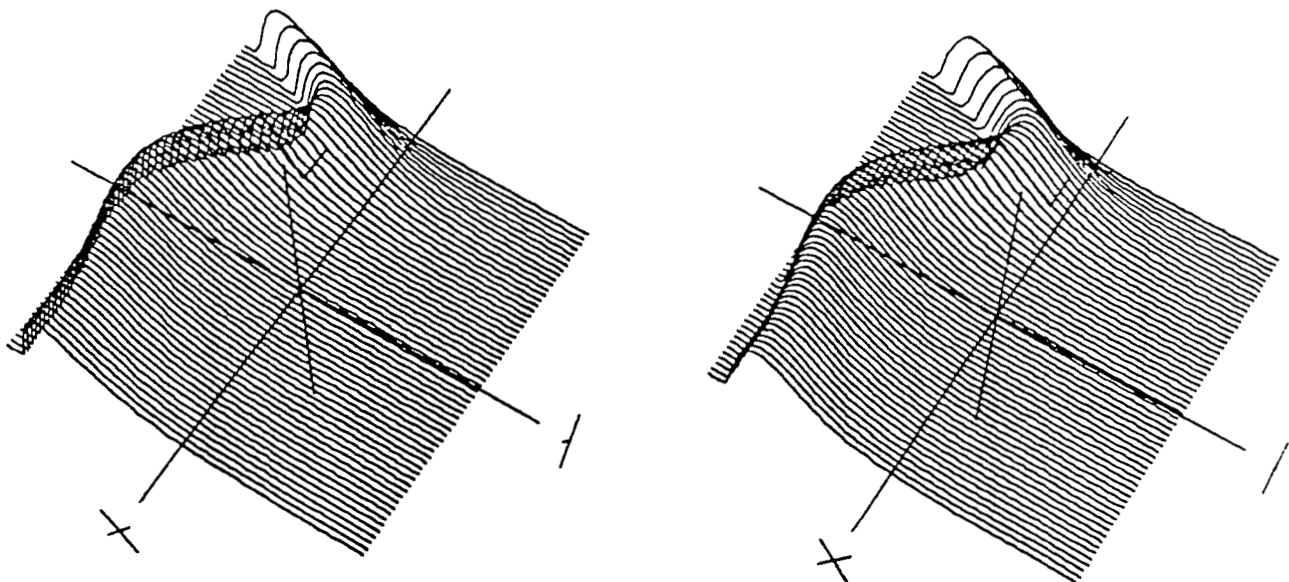


Figure 4a. Integrand in equation (15) for gamma distribution on damped sine. The X-axis represents the variable  $t$  and the Y-axis the variable  $Z$ .

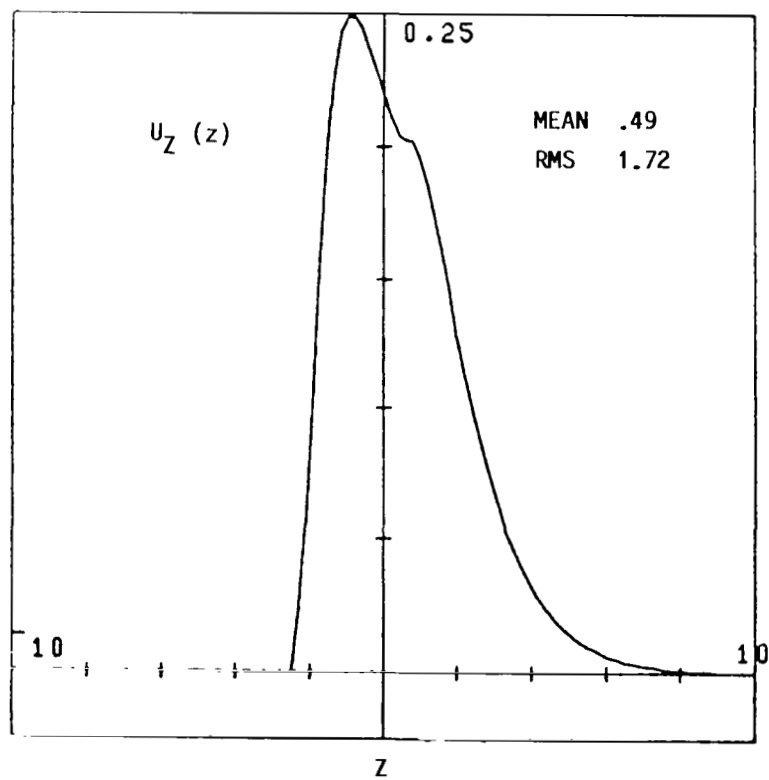


Figure 4b. Probability density function for gamma noise on damped sine wave.

## REFERENCES

1. Gnedenko, B. V.: The Theory of Probability. Chelsea Publishing Company, N.Y., 1967, p. 277-279.
2. Abramowitz, M. and Stegun, I. A.: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. National Bureau of Standards, Applied Mathematics Series, 55, U. S. Government Printing Office, Washington, 1964, p. 360.

## APPENDIX A

### STEREO VISUALIZATION WITHOUT OPTICAL AIDS<sup>3</sup>

#### (Cross-Eyed Stereo)

On many occasions in engineering and physical analysis, it would be useful to be able to sketch in three dimensions. To fulfill this wish in many cases, a convenient technique which requires no optical devices other than one's eyes may be used. All that is required is a stereoscopic pair of images. One additional capability is necessary. The observer must be able to cause the lines of sight of his eyes to converge; i.e., one must cross one's eyes. The stereo projections are formed as shown in Figure A-1. The images are reversed and viewed as in Figure A-2. With a little practice, one can easily learn to reconstruct mentally, the 3-dimensional scene from the reversed stereo pairs. In this report there are two stereo pictures. The interested reader should try several viewing distances. (The farther away the page the less crossing of the eyes is required and the easier it becomes to focus the images.) Squinting may also help as it increases one's depth of focus. When one first looks at a stereo pair, one focuses on the page and sees two similar but separate images. As one begins to cross his eyes, the two images become four. Continue crossing the eyes until the interior pair of images come together. Since the line connecting corresponding points on the images must be at the same angle about the line of sight as the line connecting the eyes, it may be necessary to rotate the page or rock the head until these two images become superimposed and seem to merge into a stereo image. This technique does require some practice but once mastered it can be very useful for easy visualization in 3 dimensions.

If a computer with plot capability is available, we can construct the necessary stereo projections from a set of points and lines that represent the object of interest. We have referred to such a representation as a wire frame model because of the appearance of the image. Let  $P$  be a representative point of the model. Each point  $P$  is projected into the picture plane  $S$  as shown in Figure A-3. The point  $P$  is projected to the eyepoint  $E$  and the line  $PE$  intersects the picture plane  $S$  at  $P'$ .  $P'$  is the projection of  $P$  onto  $S$ . The set of all points  $P'$  projected from object points  $P$  together with the connecting lines form the desired projection. We set up a reference frame in the plane  $S$ . To do this, we must specify which way is up (so to speak). Let  $\underline{u}_u$  be a unit vector in this direction and  $\underline{u}_r = \underline{u}_u \times \underline{\ell}$  is a unit vector in  $S$  pointing to the right. We place the origin of the  $S$  coordinate reference at  $O$ . Observe that  $\underline{r}_O = \underline{r}_E + d \underline{\ell}$ ; where  $\underline{r}_O$  and  $\underline{r}_E$  are position vectors of  $O$  and  $E$ , respectively. From the geometry shown in Figure A-3, we can see that

$$\underline{r}_{P'} = \underline{r}_E + (\underline{r}_P - \underline{r}_E) d / \underline{\ell} \cdot (\underline{r}_P - \underline{r}_E) \quad .$$

From this we can compute

$$x_{P'} = (\underline{r}_{P'} - \underline{r}_O) \cdot \underline{u}_r$$

$$y_{P'} = (\underline{r}_{P'} - \underline{r}_O) \cdot \underline{u}_u \quad .$$

Since  $\underline{\ell} \cdot \underline{u}_u = \underline{\ell} \cdot \underline{u}_r = 0$ ,

3. Extracted from NASA TM-78252, Torque Equilibrium Attitude Control For Skylab Reentry, Nov. 1976, by John R. Glaese and Hans F. Kennel.



$$x_{P'} = (r_{P'} - r_E) \cdot u_r$$

$$y_{P'} = (r_{P'} - r_E) \cdot u_u \quad .$$

The set of points  $(x_{P'}, y_{P'})$  plotted conventionally forms the desired projection. Size can be altered by scale adjustments. These projections are then placed as desired. Also, the values used for  $d$  and eye separation  $s$  are arbitrary and can be adjusted for convenience or eye comfort. In real life  $s \cong 65$  mm and  $d \cong 250$  mm for comfortable reading; however, it may be more comfortable for  $d$  to be larger. Some initial experimentation with this technique should establish desirable settings.

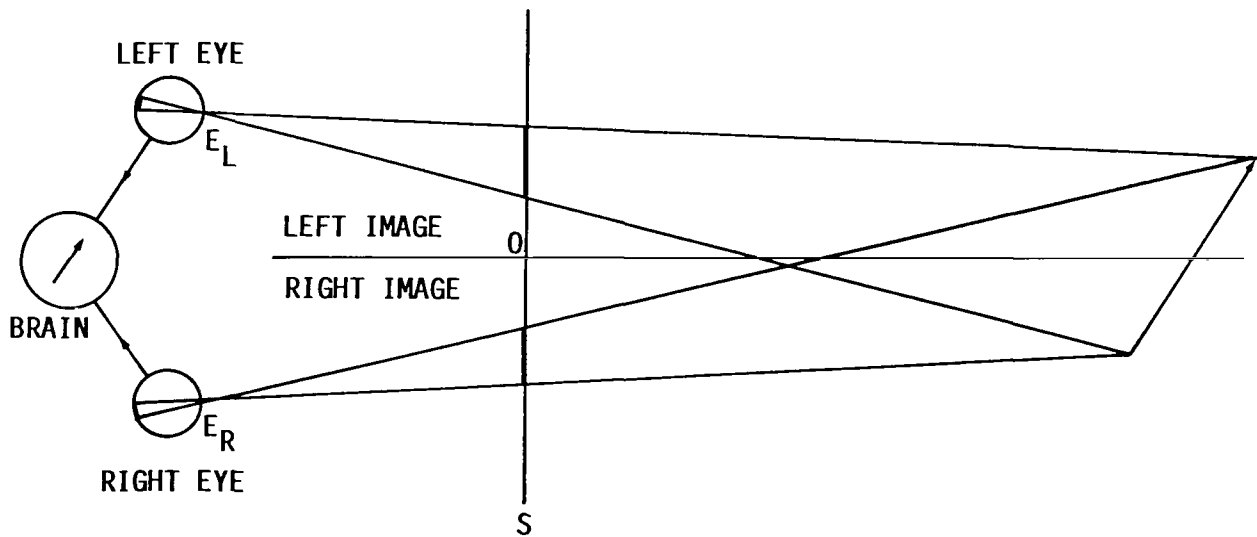


Figure A-1. Stereo projection.

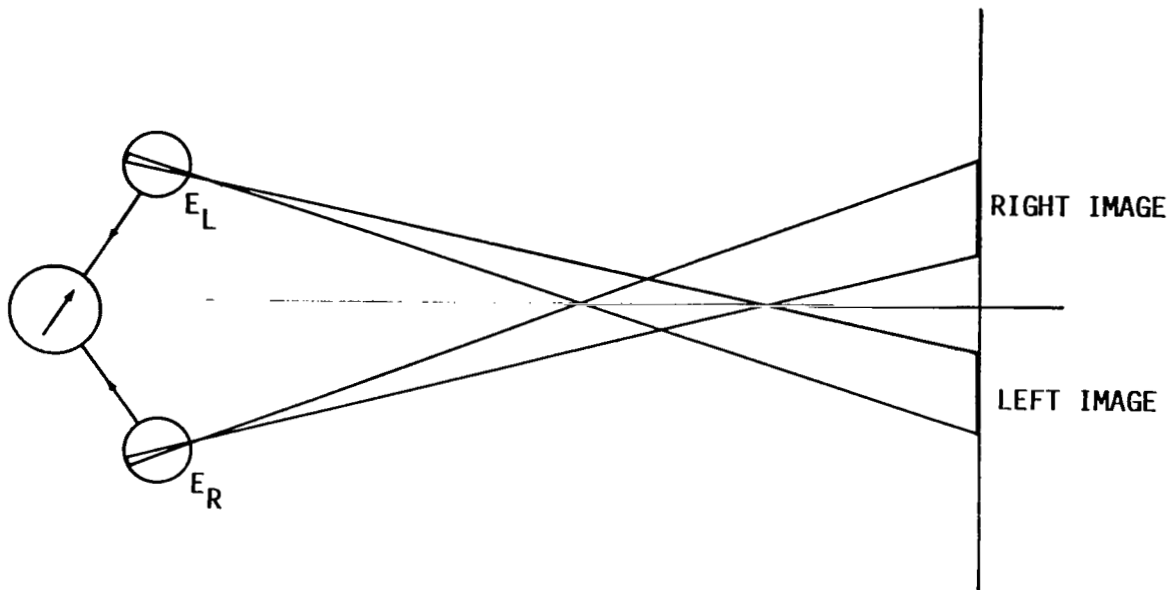


Figure A-2. Stereo reconstruction by cross-eyed viewing.

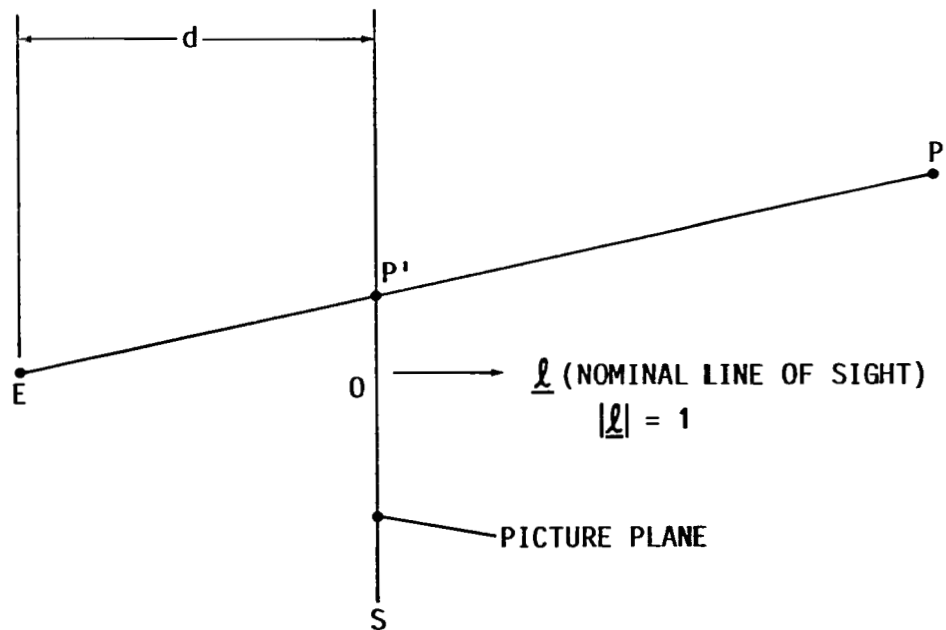


Figure A-3. Projection geometry.

1. REPORT NO. NASA TP-2216		2. GOVERNMENT ACCESSION NO.		3. RECIPIENT'S CATALOG NO.	
4. TITLE AND SUBTITLE Analysis of Random Signal Combinations for Spacecraft Pointing Stability				5. REPORT DATE August 1983	
7. AUTHOR(S) Leonard Howell				6. PERFORMING ORGANIZATION CODE	
9. PERFORMING ORGANIZATION NAME AND ADDRESS George C. Marshall Space Flight Center Marshall Space Flight Center, Alabama 35812				8. PERFORMING ORGANIZATION REPORT #	
12. SPONSORING AGENCY NAME AND ADDRESS National Aeronautics and Space Administration Washington, D.C. 20546				10. WORK UNIT, NO. M-420	
				11. CONTRACT OR GRANT NO.	
				13. TYPE OF REPORT & PERIOD COVERED Technical Paper	
15. SUPPLEMENTARY NOTES Prepared by Systems Dynamics Laboratory, Science and Engineering Directorate.				14. SPONSORING AGENCY CODE	
16. ABSTRACT  Methods for obtaining the probability density function of random signal combinations are discussed. These methods provide a realistic criteria for the design of control systems subjected to external noise with several important applications for aerospace problems.					
17. KEY WORDS  Probability density function, Space Telescope, control system, convolutions, characteristic function, random variable			18. DISTRIBUTION STATEMENT  Unclassified - Unlimited  Subject Category 65		
19. SECURITY CLASSIF. (of this report) Unclassified		20. SECURITY CLASSIF. (of this page) Unclassified		21. NO. OF PAGES 16	
				22. PRICE A02	

National Aeronautics and  
Space Administration

Washington, D.C.  
20546

Official Business

Penalty for Private Use, \$300

THIRD-CLASS BULK RATE

Postage and Fees Paid  
National Aeronautics and  
Space Administration  
NASA-451



4 1 10, G, 830817 500903DS  
DEPT OF THE AIR FORCE  
AF WEAPONS LABORATORY  
ATTN: TECHNICAL LIBRARY (SUL)  
KIRTLAND AFB NM 87117

**NASA**

POSTMASTER: If Undeliverable (Section 158  
Postal Manual) Do Not Return

---